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Coherent Motion with Linear Coupling

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Abstract

Coupling of the transverse degrees of freedom modifies analysis of beam coherent motion. A general, simple and effective rule to do that is derived.

1 Introduction

When the two fractional tunes of a storage ring are equal, $\{\nu_x\} = \{\nu_y\}$, the beam stays at a coupling resonance. This line in the tune space is of a special attraction: staying there maximizes tune area free from the dangerous resonances. That is why so many machines stayed, stay or plan to be there. Near the coupling resonance, even a small skew quadrupole or solenoid may result in a significant change of the beam optics, making it strongly coupled. If so, a conventional uncoupled 2D optical formalism cannot be used; instead, a 4D analysis has to be applied. Thus, any beam issue underlain by the optics has to be revisited, assuming that eigenmodes do not describe planar vertical and horizontal motion any more. One of these issues is a problem of the beam transverse coherent motion. This problem was considered in Refs. [1], [2], and discussed in [3]. Here, we suggest our view of the problem, and come to a solution, which is general and simple at the same time. The leading idea is that the classical mechanics is invariant over the canonical transformations. In a basis of the eigenmodes, the beam motion gets to be uncoupled, and formally similar to conventional $x-y$ uncoupled case. There is though a single difference between the $x-y$ space and the space of the normal modes. This difference relates to wake functions or impedances, which are given in the $x-y$ space. Thus, to solve the problem, the wakes and impedances have to be properly projected on the eigenvectors.

2 Eigenmodes perturbations

For arbitrary coupling, the beam optics can be described in terms of 4D eigenvectors. Hereafter, a parametrization suggested in [4] is used, where the 4 eigenvectors $\mathbf{V}_1,\ \mathbf{V}_{-1} \equiv \mathbf{V}_1^*,\ \mathbf{V}_2,\ \mathbf{V}_{-2} \equiv \mathbf{V}_2^*$ of a revolution matrix $\mathbf{R}$ are presented as follows:

$$\mathbf{V}_1 = \left(\sqrt{\beta_{1x}}\ ,\quad -\frac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}}\ ,\quad \sqrt{\beta_{1y}}e^{i\nu_1}\ ,\quad -\frac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}}e^{i\nu_1}\right)^{\mathrm{T}} \ ; \qquad (1)$$

$$\mathbf{V}_2 = \left(\sqrt{\beta_{2x}}e^{i\nu_2}\ ,\quad -\frac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}}}e^{i\nu_2}\ ,\quad \sqrt{\beta_{2y}}\ ,\quad -\frac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}}\right)^{\mathrm{T}} \ ; \qquad (2)$$

where the superscript $^{\mathrm{T}}$ stands for the transposed form, and $\mathbf{R} \cdot \mathbf{V}_m^{(0)} = \exp(-i\mu_m)\mathbf{V}_m^{(0)}$. Components of the 4D vectors are transverse coordinates and angles, $(x, \theta_x, y, \theta_y)$; in case of non-zero longitudinal magnetic field, the angles are modified according to a conventional rule for the canonical momenta [4]. Eigenvector parameters β_{1x}, β_{2y}, etc. are determined by the machine optics. The symplecticity requires then a specific orthogonality

$$\mathbf{V}_m^+ \cdot \mathbf{U} \cdot \mathbf{V}_n = -2i\delta_{mn}\mathrm{sgn}(m) \ ; \qquad (3)$$

with the superscript $^+$ meaning Hermite-conjugation, δ_{mn} is the Kronecker symbol, $\mathrm{sgn}(m)$ is the sign function, and

$$\mathbf{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad (4)$$

is the symplectic unit matrix. This formalism is in fact an extension of Ripken-Mais presentation [5], and is closely related to the Edwards-Teng parametrization [6]. Any vector $\mathbf{X}$ in the 4D phase space can be expanded over the eigenvectors (1):

$$\begin{aligned} \mathbf{X} &= \sum_n C_n \mathbf{V}_n \ ; & (5) \\ C_n &= \frac{i}{2}\mathbf{V}_n^+ \cdot \mathbf{U} \cdot \mathbf{X} \ ; \quad C_{-n} = C_n^* \ ; \ (n > 0) & \end{aligned}$$

Now, an elementary act of two-particle interaction has to be considered in terms of the eigenmodes. Following A. Chao's notations, [7], the elementary kick for angles of the following particle $(\Delta\theta_x, \Delta\theta_y)$ is expressed as

$$\begin{aligned} \Delta\theta_x &= -e^2 x W_x/(p_0 v_0) \ ; & (6) \\ \Delta\theta_y &= -e^2 y W_y/(p_0 v_0) \ . & \end{aligned}$$

Here e is the particle charge, p_0 and v_0 are a longitudinal velocity and momentum in the laboratory frame, x and y are the offsets, and $W_{x,y}$ are the wake functions. In terms of the 4D vector $\mathbf{X} = (x, \theta_x, y, \theta_y)$, this can be expressed as a perturbation $\Delta\mathbf{X} = \mathbf{W} \cdot \mathbf{X}$ with the wake matrix elements $\mathbf{W}_{2,1} = -e^2 W_x/(p_0 v_0)$,

$\mathbf{W}_{4,3} = -e^2 W_y/(p_0 v_0)$, and zeros for all other matrix elements. In terms of the complex amplitudes C_n (5), this kick is expressed as

$$\Delta C_n = \frac{i}{2}\mathbf{V}_n^+ \cdot \mathbf{U}{\cdot}\Delta\mathbf{X} = \frac{i}{2}\sum_m \mathbf{V}_n^+ \cdot \mathbf{U} \cdot \mathbf{W} \cdot \mathbf{V}_m C_m \equiv \frac{i}{2}\sum_m \mathbf{G}_{n,m} C_m \ . \tag{7}$$

The kick matrix $\mathbf{G}$ is not diagonal generally; so, when the mode m is originally excited, the wake drives other modes $n \neq m$ as well. However, when the wake is small enough, it can be treated as a small perturbation of the coherent eigenmode amplitudes. In this case, in the first order of the perturbation theory, only diagonal elements of the perturbation are important, similar to the Quantum Mechanics (see in more details [8]). The wake mixing can be considered as small in this sense, when the tune separation of the two transverse modes is much bigger than the wake-driven coherent tune shift:

$$|\nu_1 - \nu_2| \gg \Delta\nu_{\rm coh} \ . \tag{8}$$

In reality, this condition is typically satisfied. If it is not, non-diagonal elements of the kick matrix $\mathbf{G}$ have to be taken into account as well. The diagonal elements are calculated as follows:

$$G_n \equiv \mathbf{G}_{n,n} = \mathbf{G}_{n,-n} = -\frac{e^2}{p_0 v_0}(W_x \beta_{nx} + W_y \beta_{ny}) \ ; \ (n = 1, 2). \tag{9}$$

This result already shows how the wake is projected on the eigenmodes. However, one more step may be useful for understanding. The complex amplitudes C_n can be presented with explicitly written real and imaginary parts as

$$C_n = \frac{q_n}{2} + i\frac{p_n}{2} \ ; \ (n = 1, 2). \tag{10}$$

It is straightforward to show that a linear phase space transformation from the original variables $(x, \theta_x, y, \theta_y)$ to the new variables (q_1, p_1, q_2, p_2) is canonical, since they are related to each other by a symplectic matrix, composed from real and imaginary parts of the eigenvectors $\mathbf{V}$ (see Ref. [4]). Thus, q_1, q_2 are new canonical coordinates, and p_1, p_2 are the corresponding canonical momenta. It follows then, that a single excited mode gets the wake-driven kick with

$$\begin{aligned} \Delta q_n &= 0 \ ; \\ \Delta p_n &= G_n q_n = -\frac{e^2}{p_0 v_0}(W_x \beta_{nx} + W_y \beta_{ny}) q_n \ ; \ (n = 1, 2). \end{aligned} \tag{11}$$

Equations (11) show how canonical momentum is perturbed by a small localized wake. Having that, the Vlasov equation with all its results in the phase space (q_1, p_1) are exactly identical to the uncoupled case (x, θ_x), with the following substitution rules for the tune $\nu_x = \mu_x/(2\pi)$, wake times beta-function $W_x \beta_x$, and, thus, impedance times beta-function $Z_x \beta_x$:

$$
\begin{aligned}
\nu_x &\rightarrow \nu_1 \; ; \\
W_x\beta_x &\rightarrow W_x\beta_{1x} + W_y\beta_{1y} \; ; \\
Z_x\beta_x &\rightarrow Z_x\beta_{1x} + Z_y\beta_{1y} \; .
\end{aligned} \tag{12}
$$

Note that these rules work both for coasting or bunched beam, and do not depend on a shape of the longitudinal potential well. Any solution of the Vlasov equation for an uncoupled beam can be immediately re-written to the coupled case with these simple rules. After that, the result looks formally similar, while its practical consequences are generally different because of two reasons. First, the incoherent betatron spectrum is changed by the coupling, $\nu_x \rightarrow \nu_1$; thus, the Landau damping is changed. This point is missed in Refs. [1], [2], where denominators of dispersion integrals are based on the uncoupled incoherent tunes. And second, an amplitude of the coherent shift $\propto Z_x\beta_{1x} + Z_y\beta_{1y}$ is a function of coupling as well. The wake substitution rule (12) is valid both for conventional driving (or dipole) wake, and for the detuning (quadrupole) wake (about the two wakes see e. g. Ref. [9]).

3 Conclusion

A method to treat $x - y$ coupling for analysis of beam transverse coherent oscillations is presented. As soon as the coherent tune shift is small compared with tune separation of the two eigenmodes, solution of the Vlasov equation for the coupled case is immediately obtained from the corresponding uncoupled solution with simple substitution rules.

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